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ORIGINAL PAPER

Hashin–Shtrikman bounds on the effective thermal conductivity of a transversely isotropic two-phase composite material

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Abstract This paper is concerned with the estimation of the effective thermal conductivity of a transversely isotropic two phase composite. We describe the general construction of the Hashin–Shtrikman bounds from first principles in the conductivity setting. Of specific interest in composite design is the fact that the shape of the inclusions and their distribution can be specified independently. This case covers a multitude of composites used in applications by taking various limits of the spheroid aspect ratio, including both layered media and unidirectional composites. Furthermore the expressions derived are equally valid for a number of other effective properties due to the fact that Laplace's equation governs a significant range of applications, e.g. electrical conductivity and permittivity, magnetic permeability and many more. We illustrate the implementation of the scheme with several examples.

Keywords Hashin–Shtrikman bounds · Conductivity · Transport problem · Hill tensor

Mathematics Subject Classification 74Q20 · 74Q15

1 Introduction

The determination of effective physical properties of heterogeneous materials obtained by mixing different phases, usually on a very small scale denoted by $\eta > 0$ is a widely

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studied problem in the physical sciences. Motivation comes from a number of areas, e.g. prediction of the overall behavior of ceramics or superconducting fibre-reinforced materials. Predicting exact fields is difficult due to the size of the microstructure. It could also be argued that such a precise solution is unnecessary if one is only interested in overall behaviour on the macroscale. A number of different techniques have been devised to determine effective properties. One such method is asymptotic homogenization theory, which involves taking the limit $\eta \to 0$, providing a homogenized governing boundary value problem with constant coefficients, see e.g. Tartar [1,2] for the N-dimensional case, or in a two dimensional setting by Lurie and Cherkaev [3].

In addition to this approach, there is also the so-called *micromechanics* community, who are often concerned with determining approximations and bounds on effective properties. This information is often very useful from a practical viewpoint. Bounds are determined via variational principles and have been studied extensively by many authors, e.g. [4–8]. The quasi-static *transport problem* (e.g. electrical and thermal conductivity, etc) is of great interest in many applications. In the conductivity setting, the Maxwell principle for the conductivity of a host material containing a suspension of spheres, is very well-known [9]. When the only information known regarding the microstructure is the volume fraction ϕ_r and the conductivity tensor $\mathbf{K}^r$ (with Cartesian components K^r_{ij}) associated with the rth phase, $r = 0, 1, \ldots, n$, the effective conductivity tensor $\mathbf{K}^*$ associated with an arbitrary medium (isotropic or anisotropic) can be estimated by the Wiener bounds for the transport problem [10], as follows

$$K^-_{ij} \leq K^*_{ij} \leq K^+_{ij}, \qquad (K^-)^{-1}_{ij} = \sum_{r=0}^{n} \phi_r (K^r)^{-1}_{ij}, \qquad K^+_{ij} = \sum_{r=0}^{n} \phi_r K^r_{ij}.$$

where K^{-1}_{ij} denotes the inverse of the tensor $\mathbf{K}$.

The Wiener bounds depend only on the phase volume fraction (and phase material properties) but are independent of any other characteristics of the microstructure. Therefore, generally they are too far apart to be of any predictive interest except at either very low or very high volume fractions, where other methods can be of great use in any case. Using a variational principle and incorporating first order statistical information such as the correlation function (note that volume fraction is usually known as zeroth order statistical information), better bounds were obtained by Hashin and Shtrikman [4], who provided the tightest possible range of variation for the property under study, knowing only volume fraction and macroscopic anisotropy. In the case of a statistically isotropic two-phase composite ($K^r_{ij} = \kappa_r \delta_{ij}, r = 0, 1$ where δ_{ij} denotes the second order identity tensor), the Hashin–Shtrikman (HS) bounds for the effective thermal conductivity $K^*_{ij} = \kappa_* \delta_{ij}$ ($\kappa^-_* \leq \kappa_* \leq \kappa^+_*$) are given by

$$\kappa^-_* = \frac{\kappa_0\kappa_1 + 2\kappa_0(\kappa_0\phi_1 + \kappa_1\phi_0)}{2\kappa_0 + \kappa_0\phi_0 + \kappa_1\phi_1}, \quad \kappa^+_* = \frac{\kappa_0\kappa_1 + 2\kappa_1(\kappa_0\phi_1 + \kappa_1\phi_0)}{2\kappa_1 + \kappa_1\phi_1 + \kappa_0\phi_0}.$$

Derivations of the HS bounds have been improved and revised by many authors since they were originally devised [5,6]. In particular by introducing a comparison material and incorporating additional microstructural information represented by a two-point correlation function, Ponte Castañeda and Willis [7] derived a more general expression for the bounds that depends on the distribution of the inclusions as well as their shape but focussed on the case of elastostatics and also always chose the comparison phase as the host medium so that approximations rather than bounds arise.

In general, bounds that appear in the literature are almost always merely "stated" (not derived) and it is often unclear how to construct them when the material is not of simple type (e.g. isotropic spheres inside an isotropic host phase). Furthermore, discussion of how the distribution tensor affects the Hashin Shtrikman bounds for the transport problem does not appear to have been studied in any detail, in contrast to elastostatics where some studies have taken place [7]. For this reason, the objective of this work is to illustrate a direct way of constructing the HS bounds for the thermal conductivity of transversely isotropic (TI) composites from first principles incorporating information about the, perhaps non-isotropic, distributions of inclusions. That is, given the phase volume fractions and thermal conductivity, the shapes of the inclusion phases and their spatial distribution, we construct a procedure by which the HS bounds can be obtained in a straightforward manner defining the correct tensor basis set and the appropriate expressions for the Hill tensors. In particular in this respect, assuming homogeneous temperature conditions in the far field and by using the associated Green's tensor, we exploit the uniformity of the Hill tensor and the known explicit expressions for spheroidal inclusions and distributions. This formulation should be of a great utility for engineers and material scientists who may wish to construct these kinds of expressions for a variety of such media.

The paper is organized as follows. In Sect. 2 we introduce the basic formulation of the two-phase problem. Following this, in order to obtain explicit expressions for the tensors that appear in the general scheme, we make use of the so-called single inclusion problem related to the Eshelby conjecture regarding isolated inclusions. Then, in Sect. 4 we describe the general formulation of the HS bounds, initially for the general multiphase case before restricting attention to two phases. We specialize in Sect. 5 to the case of macroscopically TI materials. This specialization therefore motivates the definition of a TI second order tensor basis set. In Sect. 6 we illustrate the implementation of the construction with some examples where we analyze the influence on the effective conductivity of the different characteristics of the microstructure. We conclude in Sect. 7.

2 Context of the problem

Consider a two-phase composite material occupying a domain $\Omega \subset \mathbb{R}^3$. The macroscopic behaviour of the composite is strongly influenced by the geometric arrangement of the host phase Ω_0 and the inclusion phase Ω_1 satisfying $\Omega_0 \cup \Omega_1 = \Omega$, with respective (second order) thermal conductivity tensors denoted by $\mathbf{K}^0, \mathbf{K}^1$. We assume constant volume fractions ϕ_0 and ϕ_1 of each phase, defined by $\phi_0 = |\Omega_0|/|\Omega| \in (0, 1)$ and $\phi_1 = 1 - \phi_0 = |\Omega_1|/|\Omega|$ respectively, where $|\cdot|$ denotes a volume. The problem governing the steady state of the temperature $T \in H^1(\Omega)$ is given by the following linear elliptic equation

$$-\mathrm{div}\,(\mathbf{K}(x)\nabla T) = f \quad \text{in } \Omega \tag{2.1}$$

where we have denoted by $f \in H^{-1}(\Omega)$ the internal source term and $\mathbf{K}$ is the thermal conductivity of the composite defined by

$$\mathbf{K}(x) = \begin{cases} \mathbf{K}^0(x) & \text{if } x \in \Omega_0, \\ \mathbf{K}^1(x) & \text{if } x \in \Omega_1. \end{cases}$$

The usual continuity conditions on the matrix-inclusions interface for temperature and for the heat flux density are satisfied, i.e.:

$$T|_{\partial\Omega_0} = T|_{\partial\Omega_1}, \qquad \mathbf{K}^0\nabla T \cdot \boldsymbol{\nu}^0 = \mathbf{K}^1\nabla T \cdot \boldsymbol{\nu}^1 \text{ on } \partial\Omega_0 \cap \partial\Omega_1,$$

where $\boldsymbol{\nu}^r$ is the outward normal unit vector to $\partial\Omega_r$, $r = 0,\ 1$ (then $\boldsymbol{\nu}^0 = -\boldsymbol{\nu}^1$). We will assume that the two materials are TI, each with the x_1x_2 plane as the plane of isotropy, so that we can write

$$K^r_{ij} = \kappa_r(\Theta_{ij} + \alpha_r\delta_{i3}\delta_{j3}), \quad r = 0,\ 1, \tag{2.2}$$

where we have introduced α_r as the measure of anisotropy, with $\alpha_r = 1$ for an isotropic medium. Furthermore $\Theta_{ij} = \delta_{ij} - \delta_{i3}\delta_{j3}$, where $\boldsymbol{\delta}_{ij}$ is the second-order unit tensor.

It is usual to characterize a composite by its *macroscopic effective properties*, represented by a second order conductivity tensor $\mathbf{K}^*$ giving the linear relationship between the body averages of heat flux and the thermal gradient as follows

$$\bar{\mathbf{q}} = \mathbf{K}^*\bar{\mathbf{e}}, \qquad \mathbf{e}(\mathbf{x}) = \nabla T(\mathbf{x}),$$

where $\bar{\boldsymbol{\xi}} = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \boldsymbol{\xi}\, d\mathbf{x}$ denotes the volume average of a given local field $\boldsymbol{\xi}$. Following Hill [11] the average energy density for the composite is defined by

$$W^*(\bar{\mathbf{e}}) = \min_{\mathbf{e}\in\mathscr{E}} \frac{1}{|\Omega|}\int_\Omega W(\mathbf{x}, \mathbf{e})\, d\mathbf{x} = \frac{1}{2}\bar{\mathbf{e}} \cdot \bar{\mathbf{q}} = \frac{1}{2}\bar{\mathbf{e}} \cdot \mathbf{K}^*\bar{\mathbf{e}}. \tag{2.3}$$

where $\mathscr{E}$ is the set of admissible temperature gradients given by

$$\mathscr{E} = \{\mathbf{e} : \text{ there exists } T \in H^1(\Omega) \text{ such that } \mathbf{e} = \nabla T, T = \bar{T} \text{ on } \partial\Omega\}.$$

We will need the following definitions

Definition 1 A composite is statistically homogeneous if p^r the probability density of finding an inclusion of type r centered at a point $\mathbf{x}$, is a constant equal to $p^r = n^r/|\Omega|$ with n^r equal to the number of inclusions of type r.

Definition 2 A composite has ellipsoidal symmetry [6] for the distribution of the inclusions if the conditional probability density function $p^{s|r}(\mathbf{x}', \mathbf{x})$ for finding an inclusion of type s centered at $\mathbf{x}'$ given that there exists an inclusion of type r centered at point $\mathbf{x}$, depends on $\mathbf{x}'' = \mathbf{x}' - \mathbf{x}$ only through the expression $|A^{(rs)}_d\mathbf{x}''|$, for some matrix $A^{(rs)}_d$ which defines an ellipsoid $\omega^{(rs)}_d = \{\mathbf{z} : \quad |\mathbf{A}^{(rs)}_d\mathbf{z}| < 1\}$.

3 The single inclusion problem: the Hill tensor for TI isotropic materials

The so-called single inclusion problems are well known and utilized in micromechanics because they allow one to derive very useful explicit expressions for the effective properties of heterogeneous materials [12]. Consider the domain Ω infinitely extended in $\mathbb{R}^3$. Assuming the inclusion is an ellipsoidal inhomogeneity ω embedded in the host matrix Ω (with thermal conductivity tensors $\mathbf{K}^1$ and $\mathbf{K}^0$ respectively), there exists the fundamental property of uniformity of thermal gradients interior to the ellipsoid under homogeneous far-field conditions (see [9,13]). An analogous property for the elasticity setting was obtained by Eshelby [14] in the case of isotropic host phases. Effective properties of inhomogeneous media can often be written rather conveniently in terms of the Eshelby or Hill tensors as a result of these useful uniformity properties.

In this section, we derive the so-called Hill P-tensor associated with the single inclusion problem by using a direct approach based on the variational formulation for the transport equation. The Green tensor associated with the matrix phase is denoted by G and satisfies

$$-\mathrm{div}\left[\mathbf{K}^0 \nabla G(\mathbf{x}-\mathbf{y})\right] = \delta(\mathbf{x}-y)$$

where $G(\mathbf{x}-\mathbf{y})$ is the Green function giving the temperature at point $\mathbf{y}$ generated by a unit point heat source at point $\mathbf{x} \in \Omega$, where $\delta(\mathbf{z})$ the n-dimensional Dirac delta function. The temperature distribution can be stated in integral form as follows

$$T(\mathbf{y}) = T^0(\mathbf{y}) - \int_\Omega G(\mathbf{x}-\mathbf{y}) f(\mathbf{x}) d\mathbf{x} - (\mathbf{K}^1 - \mathbf{K}^0) \int_\omega \nabla T(\mathbf{x}) \nabla G(\mathbf{x}-\mathbf{y}) d\mathbf{x}, \tag{3.4}$$

$\forall \mathbf{y} \in \Omega$, where T^0 is the solution of (2.1) without any inclusion present ($\mathbf{K}^0 = \mathbf{K}^1$). Neglecting the influence of the source term f (since this should not affect effective properties) and taking homogeneous temperature conditions in the far field, the expression (3.4) becomes

$$\mathbf{e}_\omega = \mathscr{C}_\omega \bar{\mathbf{e}}, \qquad \mathscr{C}_\omega = \left[\boldsymbol{I} + \mathbf{P}(\mathbf{K}^1 - \mathbf{K}^0)\right]^{-1} \tag{3.5}$$

where $I_{ij} = \delta_{ij}$. This expression relates the uniform temperature gradient inside the inclusion ω to the average temperature gradient inside the medium Ω through the so-called concentration tensor $\mathscr{C}_\omega$. In (3.5), $\mathbf{P}$ is known as the Hill-tensor (also called the P-tensor) which following (3.4) is given by

$$P_{ij} = -\frac{\partial^2}{\partial y_i \partial y_j} \int_\omega G(\mathbf{x}-\mathbf{y}) d\mathbf{x}, \quad \forall y \in \Omega. \tag{3.6}$$

In particular, the components for the uniform Hill-tensor for an ellipsoidal inclusion with semi-axes a_j, $j = 1, 2, 3$ are given by see [15]

$$P_{ij} = \frac{\det(\mathbf{A})}{4\pi} \int_S \frac{\boldsymbol{\Gamma}^0_{ij}(\boldsymbol{\xi})}{(\boldsymbol{\xi}^T(\mathbf{A})^T\mathbf{A}\boldsymbol{\xi})^{\frac{3}{2}}}\, dS \tag{3.7}$$

where the superscript T denotes transpose, S denotes the surface of the unit sphere and $\mathbf{A}$ and $\boldsymbol{\Gamma}^0$ are second order tensors with components

$$A_{ij} = \sum_{k=1}^{3} a_k \delta_{ik}\delta_{jk} \quad \text{and} \quad \boldsymbol{\Gamma}^0_{ij}(\boldsymbol{\xi}) = \frac{\xi_i \xi_j}{K^0_{kl}\xi_k\xi_l}.$$

Although the expression (3.7) can be applied to more general geometries (ellipsoids that are aligned or non-aligned with axes of anisotropy), for the sake of simplicity in this work we will consider oblate or prolate spheroidal TI inclusions with semi-axes $a = a_1 = a_2 \neq a_3$ inside a transversely isotropic matrix material, where the a_3 semi-axis is aligned with the x_3 axis of transverse isotropy of the matrix phase. In this case, the components of $\mathbf{P}$ are given by (see [15] for details)

$$P_{ij} = \frac{1}{\kappa_0}(\varphi\Theta_{ij} + \varphi_3\delta_{i3}\delta_{j3}),\ \varphi = \frac{1}{2}(1-\alpha_0\varphi_3),\ \varphi_3 = \frac{1}{\alpha_0}\mathscr{S}\left(\frac{\varepsilon}{\sqrt{\alpha_0}}\right), \tag{3.8}$$

where $\varepsilon = a_3/a$ is the aspect ratio of the *spheroidal* inclusion and the function $\mathscr{S}$ is given by the expression

$$\mathscr{S}(x) = \frac{1}{1-x^2} - \frac{x}{1-x^2} \times \begin{cases} \dfrac{1}{(x^2-1)^{\frac{1}{2}}}\operatorname{arccosh}(x),\ \forall x \in (1,+\infty) \quad \text{(problate)}, \\ \dfrac{1}{(1-x^2)^{\frac{1}{2}}}\arccos(x),\quad \forall x \in [0,1) \quad \text{(oblate)}\,. \end{cases}$$

Note that limits when $\varepsilon \to 0$ and $\varepsilon \to \infty$ correspond to particular cases of disc or layered medium and a long fibre-reinforced medium respectively. The Hill-tensor (3.8) in these cases simplify to the forms

$$P^{\text{fibre}}_{ij} = \frac{1}{2\kappa_0}\Theta_{ij}, \qquad P^{\text{layered}}_{ij} = \frac{1}{\kappa_0}\delta_{i3}\delta_{j3}. \tag{3.9}$$

If the heterogeneity has *spherical* shape ($\varepsilon = 1$), (3.8) leads to

$$P_{ij} = \frac{1}{\kappa_0}(\varphi\Theta_{ij} + \varphi_3\delta_{i3}\delta_{j3}),\ \varphi = \frac{1}{2}(1-\alpha_0\varphi_3),\ \varphi_3 = \frac{1}{\alpha_0}\mathscr{S}\left(\frac{1}{\sqrt{\alpha}}\right). \tag{3.10}$$

For the particular case of *spherical* inclusions embedded inside an *isotropic* phase ($\mathbf{K}^0 = \kappa_0 I, \mathbf{K}^1 = \kappa_1 I$), the Hill-tensor simplifies to the form $\mathbf{P} = \dfrac{1}{3\kappa_0} I$.

4 The Hashin–Shtrikman variational principle

Hashin and Shtrikman [4] derived a variational principle to bound the overall conductivity of a heterogeneous composite with *statistically isotropic* microstructure. It was based on an alternative representation of the effective energy of the heterogeneous media that makes use of a proper homogeneous comparison material (see also [5]). An extension of the Hashin–Shtrikman formulation was studied by several authors, additionally in the elasticity context (see [16]). In particular, Willis [6] developed a variational structure to obtain these classical bounds on an anisotropic composite comprising a matrix and n different types of inclusion phase. In this work, a polarization field is introduced relative to the comparison material, and the distribution tensor of inclusions is assumed the same as their shape. (given by integrals of the associated two-point correlation functions). Later, Ponte Castañeda and Willis generalized the previous structure given by Willis [6] to the case where the influence of the shape and the distribution of the different phases are taken into account independently.

In this section, following Ponte Casteñeda and Willis' [7] derivation of the variational principle for the elasticity context, we will derive explicit bounds for the effective conductive energy of an anisotropic composite, first for the multiphase-case before restricting attention to the two-phase scenario. In this formulation the hypotheses of *ellipsoidal symmetry* for the distribution of the inclusions (possibly with a different ellipsoidal shape to the shape of the inclusion) is assumed. Therefore, the explicit expressions (3.8–3.10) for the P-tensor can be used, thanks to the linearity of the problem, leading to explicit bounds on the effective conductivity $\mathbf{K}^*$.

Assume at first that the composite occupies the domain Ω comprising n different types of inclusion phases that could be selected independently of their spatial distribution depending on two parameters $\varepsilon,\ \rho > 0$ respectively. These parameters are the aspect ratio of the spheroidal inclusion (ε) and the statistics associated with the spheroidal distribution function (ρ). We denote the conductivity tensor of the rth phase by $\mathbf{K}^r$, $r = 1, \ldots, n$ and by $\mathbf{K}^0$ the conductivity of the matrix within which the inclusion phases are embedded. Ω_r represents the total domain of the rth phase, i.e. it is the collection of n^r ellipsoidal inclusions (each aligned and having the shape defined by a domain ω^r) with total volume fraction equal to $\phi_r = |\Omega_r|/|\Omega|$. The volume fraction of the host phase is therefore $\phi_0 = 1 - \sum_{r=1}^{n} \phi_r$. Under these conditions, we have the following theorem [7]

Theorem 1 *Let* $\mathbf{K}^c$ *be the uniform conductivity tensor of a homogeneous comparison material. Assume that the composite is statistically homogeneous and the distribution of inclusion phases is defined by spheroidal symmetry. Then, given* $\rho,\ \varepsilon > 0$ *its effective energy* W^* *satisfies the inequality*

$$W^*(\bar{\mathbf{e}}) \geq (\leq)\, \frac{1}{2}\bar{\mathbf{e}} \cdot \mathbf{K}^c \bar{\mathbf{e}} + \frac{1}{2}\bar{\mathbf{e}} \cdot \bar{\boldsymbol{\tau}}^* \tag{4.11}$$

whenever $\mathbf{K}^c \leq \min_{0 \leq r \leq n} \mathbf{K}^r (\geq \max_{0 \leq r \leq n} \mathbf{K}^r)$, *where* $\bar{\boldsymbol{\tau}}^* = \sum_{k=0}^{n} \phi_k \boldsymbol{\tau}_k^*$ *is the average of the optimal polarizations* $\boldsymbol{\tau}_k^*$, *which satisfy the relations*

$$
\begin{aligned}
&(\mathbf{K}^0-\mathbf{K}^c)^{-1}\boldsymbol{\tau}_0^*-\frac{1}{\phi_0}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\mathbf{M}^{(k\ell)}(\boldsymbol{\tau}_\ell^*-\boldsymbol{\tau}_0^*)=\bar{\mathbf{e}}\\
&(\mathbf{K}^k-\mathbf{K}^c)^{-1}\boldsymbol{\tau}_k^*+\frac{1}{\phi_k}\sum_{\ell=1}^{n}\mathbf{M}^{(k\ell)}(\boldsymbol{\tau}_\ell^*-\boldsymbol{\tau}_0^*)=\bar{\mathbf{e}},\quad k=1,\ldots,n.
\end{aligned}
\tag{4.12}
$$

Max and min hold componentwise. The parameters $\mathbf{M}^{(k\ell)}$ in (4.12) depend on $\mathbf{K}^0$ and on the microstructure. They can be shown to be symmetric and to have the form

$$
\mathbf{M}^{(k\ell)}=\phi_\ell(\mathbf{P}_s^{\varepsilon,k}-\phi_k\mathbf{P}_d^{\rho,(k\ell)}),\quad k,\,\ell=1,\ldots,n. \tag{4.13}
$$

Here, $\mathbf{P}_s^{\varepsilon,k}$ and $\mathbf{P}_d^{\rho,(k\ell)}$ denote the uniform Hill-tensors given by (3.7) associated with the aspect ratio ε of the inclusion in the kth phase and with the aspect ratio ρ associated with the spheroidal distribution and concerning the interaction between the kth and ℓth phases.

Proof Denote by W^c the energy function associated with the comparison material with uniform conductivity tensor $\mathbf{K}^c$ and assume that $\mathbf{K}^c\le\min_{r=1,\ldots,n}\{\mathbf{K}^r\}$ in the sense that $\mathbf{e}\cdot(\mathbf{K}^c-\mathbf{K}^r)\cdot\mathbf{e}\le 0$, $r=1,2,\ldots n$, $\forall\mathbf{e}\in\mathscr{E}$. Given $\rho,\varepsilon>0$, the Legendre-Fenchel transform for the conductivity problem is defined as

$$
(W-W^c)^o(\mathbf{x},\boldsymbol{\tau})=\max_{\mathbf{e}\in\mathscr{E}}\{\boldsymbol{\tau}\cdot\mathbf{e}-(W(\mathbf{x},\mathbf{e})-W^c(\mathbf{e}))\}. \tag{4.14}
$$

From (2.3), Eq. (4.14) gives

$$
W^*(\bar{\mathbf{e}})\ge\inf_{\mathbf{e}}\int_\Omega\{\boldsymbol{\tau}\cdot\mathbf{e}+W^c(\mathbf{e})-(W-W^c)^o(\boldsymbol{\tau},\mathbf{x})\}d\mathbf{x}. \tag{4.15}
$$

For any $\boldsymbol{\tau}$, the infimum is attained when $\mathbf{e}=\bar{\mathbf{e}}-\boldsymbol{\Gamma}\boldsymbol{\tau}$. Γ denotes the linear integral operator

$$
\big(\Gamma\boldsymbol{\tau}\big)(\mathbf{x})=\int_\Omega\Gamma^c(\mathbf{x}-\mathbf{s})(\boldsymbol{\tau}(\mathbf{s})-\bar{\boldsymbol{\tau}})d\mathbf{s}, \tag{4.16}
$$

where $\bar{\boldsymbol{\tau}}$ represent the mean value of $\boldsymbol{\tau}$ and whose kernel is related to the Green function G^c for the domain Ω with conductivity modulus tensor $\mathbf{K}^c$ given by (see [6])

$$
\Gamma_{ij}^c(\mathbf{x})=-\frac{\partial^2G^c(\mathbf{x})}{\partial x_i\partial x_j}=-\frac{1}{8\pi^2}\int_{|\zeta|=1}\mathbf{H}^c(\zeta)\delta''(\zeta\mathbf{x})dS,
$$

with $\mathbf{H}^c(\zeta)$ a tensor with components $H_{ij}^c(\zeta)=\mathbf{B}^c(\boldsymbol{\xi})\zeta_i\zeta_j$ and $\mathbf{B}^c$ the inverse of the tensor $\mathbf{C}$ with components $C_{kl}(\boldsymbol{\zeta})=\mathbf{K}^c\zeta_k\zeta_l$. Selecting a piecewise constant polarization field $\boldsymbol{\tau}(\mathbf{x})=\sum_{r=0}^n\chi_r(x)\boldsymbol{\tau}_r$ $\big($then $\bar{\boldsymbol{\tau}}=\sum_{r=0}^n\phi_r\boldsymbol{\tau}_r\big)$, following (4.16) and employing the fact that the average of $\Gamma^c\boldsymbol{\tau}$ is equal to zero, (4.15) gives

$$
\begin{aligned}
W^*(\bar{\mathbf{e}}) \geq \bar{\mathbf{e}} + \frac{1}{2}\bar{\mathbf{e}}\cdot\mathbf{K}^c\bar{\mathbf{e}} - \frac{1}{2}\sum_{r=1}^{n}\phi_r\boldsymbol{\tau}^r[(\mathbf{K}^r-\mathbf{K}^c)^{-1}\cdot\boldsymbol{\tau}^r] \\
-\frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}(\boldsymbol{\tau}^r-\boldsymbol{\tau}^1)\cdot\mathbf{M}^{(rs)}(\boldsymbol{\tau}^s-\boldsymbol{\tau}^1)
\end{aligned}
\tag{4.17}
$$

where

$$
\mathbf{M}^{(rs)} = \int_\Omega\int_\Omega \chi^r(\mathbf{x})[\chi^s(\mathbf{x}')-\phi_s]\Gamma^c(\mathbf{x}-\mathbf{x}')\,d\mathbf{x}d\mathbf{x}' \quad r,\ s=0,\ 1,\ldots,n. \tag{4.18}
$$

Inequality (4.17) is extremized with respect to the polarizations $\boldsymbol{\tau^r}$, to obtain (4.11), where $\bar{\boldsymbol{\tau}}^* = \sum_{k=0}^{n}\phi_k\boldsymbol{\tau}_k^*$ is the average of the so-called *optimal polarizations* $\boldsymbol{\tau}_k^*$ which satisfy the relationships given by (4.12).

Following [7], in order to simplify the above expression for the microstructure tensor given by (4.18), we make use of the hypotheses of *statistical homogeneity* and *spheroidal symmetry* of the composite. Namely, if the inclusions are assumed to be spheroidally distributed (but maybe with a different ellipsoid from one that defines the inclusion shapes), the tensor (4.18) can be expressed as (4.13). In (4.13) $\mathbf{P}_s^{\varepsilon,k}$ and $\mathbf{P}_d^{\rho,(k,\ell)}$ are uniform P-tensors given by (3.7) with ω replaced by the rth phase ellipsoidal inclusion ω^r and by the ellipsoid $\omega^{(k\ell)}$ given by Definition 2 respectively, and whose shapes depend on ε and ρ. □

Remark 1 Recall that the subscripts s and d refer to the *shape* and *distribution* of inclusions depending on ε and ρ respectively. By definition, we have $\mathbf{P}_d^{\rho,(k\ell)} = \mathbf{P}_d^{\rho,(\ell k)}$, and for conciseness we will write $\mathbf{P}_d^{\rho,k}$, to denote $\mathbf{P}_d^{\rho,(kk)}$, $k=0,\ldots,n$.

Corollary 1 *By linearity, using* (2.3)*, the following expression for the optimal bounds is derived from (4.11)*

$$
\mathbf{K}^* \geq (\leq)\,\mathbf{K}^c + \bar{\boldsymbol{\tau}}^*\cdot\bar{\mathbf{e}}^{-1} := \mathbf{K}^B \tag{4.19}
$$

where $\mathbf{K}^B = \mathbf{K}^+$ *for an upper bound and* $\mathbf{K}^B = \mathbf{K}^-$ *for a lower bound.*

Corollary 2 *Taking the linear comparison material with conductive modulus tensor as* $\mathbf{K}^c = \max_{0\leq r\leq n}\mathbf{K}^r$ *(resp.* $\mathbf{K}^c = \min_{0\leq r\leq n}\mathbf{K}^r$*), meaning that we choose the maximal (resp. minimal) value of each component in the tensor, by (4.19) we find that* $\mathbf{K}^B = \mathbf{K}^+$ *(resp.* $\mathbf{K}^B = \mathbf{K}^-$*) is an upper (resp. lower) bound on the effective modulus tensor. The Hashin–Shtrikman bounds are thus*

$$
\mathbf{K}^- \leq \mathbf{K}^* \leq \mathbf{K}^+.
$$

Note that $\mathbf{K}^+$ *and* $\mathbf{K}^-$ *are obtained when the comparison material are the highest and the lowest conducting phase respectively. If the comparison material is neither the highest nor the lowest conducting phase, then* $\mathbf{K}^B$ *is only an approximation to the effective properties and then, in the following it will be denoted by* $\mathscr{K}^*$.

4.1 The Hashin–Strikman bounds for a two-phase composite

In this work we shall restrict attention to two-phase particulate media so that there is a single inclusion phase. For simplicity, we suppose that the inclusion phase is a distribution of (possibly different sized) aligned spheroids but where each spheroid has the same aspect ratio $\varepsilon = a_3/a$ and where the long/short axis (same direction as the semi-axis a_3 of the spheroid) is aligned with x_3. Note that this spheroidal shape is taken into account thanks to the P-tensor $\mathbf{P}_s^\varepsilon$. Its distribution is accounted for by virtue of the P-tensor $\mathbf{P}_d^\rho$ which we shall also consider to be governed by spheroidal statistics, of aspect ratio ρ.

Various alternative expressions for the HS bounds can be obtained. Let us first consider the case when the comparison phase can be chosen as either the host or inclusion phase. Note that this may not always be possible however.

Comparison phase can be identified as either host or inclusion phase

Let us first suppose that we are able to identify $\mathbf{K}^c = \mathbf{K}^0$ (then $\boldsymbol{\tau}^0 = 0$) and then $\mathbf{K}^c = \mathbf{K}^1$ (thus $\boldsymbol{\tau}^1 = 0$). Then, from (4.11) and (4.12) the following expressions follow respectively:

$$\mathbf{K}^{B_0} = \mathbf{K}^0 + \phi_1 \left((\mathbf{K}^1 - \mathbf{K}^0)^{-1} + \mathbf{P}_s^\varepsilon - \phi_1 \mathbf{P}_d^\rho \right)^{-1}, \tag{4.20}$$

$$\mathbf{K}^{B_1} = \mathbf{K}^1 + \phi_0 \left((\mathbf{K}^0 - \mathbf{K}^1)^{-1} + \mathbf{Q}_s^\varepsilon - \phi_1 \mathbf{Q}_d^\rho \right)^{-1}, \tag{4.21}$$

where

$$\mathbf{Q}_s^\varepsilon = \frac{\phi_1}{\phi_0} \mathbf{P}_s^\varepsilon, \qquad \mathbf{Q}_d^\rho = \frac{\phi_1}{\phi_0} \mathbf{P}_d^\rho.$$

From (4.20 and 4.21) we easily derive that if the matrix is the more insulating phase (i.e. $\mathbf{K}^0 \le \mathbf{K}^1$) we have $\mathbf{K}^- = \mathbf{K}^{B_0}$ and $\mathbf{K}^+ = \mathbf{K}^{B_1}$. On the contrary, if we can identify the matrix phase as the highest conductive material (i.e. $\mathbf{K}^1 \le \mathbf{K}^0$), then $\mathbf{K}^- = \mathbf{K}^{B_1}$ and $\mathbf{K}^+ = \mathbf{K}^{B_0}$.

Remark 2 Note that expanding expression (4.20) [or analogously (4.21)] with respect to the volume fraction, we derive the following approximation for $\mathbf{K}^{B_0}$ (and therefore for $\mathbf{K}^*$)

$$\begin{aligned}\mathscr{K}^* = {} & \mathbf{K}^0 + [(\mathbf{K}^1 - \mathbf{K}^0)^{-1} + \mathbf{P}_s^\varepsilon]^{-1}\phi_1 \\ & + \left([(\mathbf{K}^1 - \mathbf{K}^0)^{-1} + \mathbf{P}_s^\varepsilon]^{-1}\phi_1 \right) \cdot \mathbf{P}_d^\rho \cdot \left([(\mathbf{K}^1 - \mathbf{K}^0)^{-1} + \mathbf{P}_s^\varepsilon]^{-1}\phi_1 \right) \\ & + \mathscr{O}((\phi_1)^3).\end{aligned} \tag{4.22}$$

From (4.22), it follows that the shape of the inclusions affect $\mathscr{K}^*$ to first order through the volume fraction, where as the corresponding to their symmetry distribution affect to second order.

Comparison material cannot be identified as either host or inclusion phase

It may not always be the case that the comparison material can be identified as exactly one of the inclusion phases. In this case both polarization fields have non-zero components and it is necessary to determine general bounds in terms of the comparison modulus tensor by solving the linear system

$$\begin{aligned}\left[(\mathbf{K}^0-\mathbf{K}^c)^{-1}+(\mathbf{Q}_s^{\varepsilon}-\phi_1\mathbf{Q}_d^{\rho})\right]\phi_0\boldsymbol{\tau}_0^*-(\mathbf{P}_s^{\varepsilon}-\phi_1\mathbf{P}_d^{\rho})\phi_1\boldsymbol{\tau}_1^*&=\phi_0\bar{\mathbf{e}},\\ -(\mathbf{Q}_s^{\varepsilon}-\phi_1\mathbf{Q}_d^{\rho})\phi_0\boldsymbol{\tau}_0^*+\left[(\mathbf{K}^1-\mathbf{K}^c)^{-1}+(\mathbf{P}_s^{\varepsilon}-\phi_1\mathbf{P}_d^{\rho})\right]\phi_1\boldsymbol{\tau}_1^*&=\phi_1\bar{\mathbf{e}}.\end{aligned}$$

which we can write as

$$\begin{aligned}\mathbf{A}_{0Q}\phi_0\boldsymbol{\tau}_0^*-\mathbf{B}_P\phi_1\boldsymbol{\tau}_1^*&=\phi_0\bar{\mathbf{e}}\\ -\mathbf{B}_Q\phi_0\boldsymbol{\tau}_0^*+\mathbf{A}_{1P}\phi_1\boldsymbol{\tau}_1^*&=\phi_1\bar{\mathbf{e}}\end{aligned}$$

so that we can approximate the effective conductive tensor by $\mathscr{K}^*$ equal to $\mathbf{K}^B$ in (4.19) and $\boldsymbol{\tau}^*$ given by

$$\begin{aligned}\boldsymbol{\tau}^*=\Big(&[\mathbf{B}_P^{-1}\mathbf{A}_{0Q}-\mathbf{A}_{1P}^{-1}\mathbf{B}_Q]^{-1}[\phi_0\mathbf{B}_P^{-1}+\phi_1\mathbf{A}_{1P}^{-1}]\\ &+[\mathbf{B}_Q^{-1}\mathbf{A}_{1P}-\mathbf{A}_{0Q}^{-1}\mathbf{B}_P]^{-1}[\phi_0\mathbf{A}_{0Q}^{-1}+\phi_1\mathbf{B}_Q^{-1}]\Big)\bar{\mathbf{e}}.\end{aligned}$$

Inclusions have the same shape as their distribution

When all the inclusions have the same shape, so that $\mathbf{P}_s^{\varepsilon,r}=\mathbf{P}_s^{\varepsilon}, r=1,\ldots,n$, and the distribution of inclusions has the same symmetry as the inclusions ($\mathbf{P}_d^{\rho}=\mathbf{P}_d^{\varepsilon}=\mathbf{P}_s^{\varepsilon}=\mathbf{P}^{\varepsilon}$), one can write down a the following clean form involving the comparison phase.

$$\begin{aligned}\mathbf{K}^B=&\left[\phi_0(\delta+(\mathbf{K}^0-\mathbf{K}^c)\mathbf{P}^{\varepsilon})^{-1}+\phi_1(\delta+(\mathbf{K}^1-\mathbf{K}^c)\mathbf{P}^{\varepsilon})^{-1}\right]^{-1}\\ &\left[\phi_0(\delta+(\mathbf{K}^0-\mathbf{K}^c)\mathbf{P}^{\varepsilon})^{-1}\mathbf{K}^0+\phi_1(\delta+(\mathbf{K}^1-\mathbf{K}^c)\mathbf{P}^{\varepsilon})^{-1}\mathbf{K}^1\right]\end{aligned}$$

This also generalizes to the multi-phase expression given by Willis [6] for composites with aligned ellipsoidal inclusions and ellipsoidal symmetry

$$\mathbf{K}^B=\left[\sum_{r=0}^{n}\phi_r\left[\delta+(\mathbf{K}^r-\mathbf{K}^c)\mathbf{P}^{\varepsilon}\right]^{-1}\right]^{-1}\sum_{r=0}^{n}\phi_r[\delta+(\mathbf{K}^r-\mathbf{K}^c)\mathbf{P}^{\varepsilon}]^{-1}\mathbf{K}^r.$$

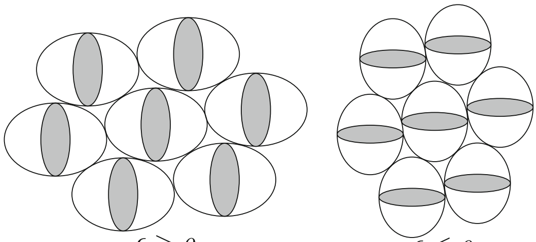

Fig. 1 *Spheroidal* distributions (aspect ratio ρ) containing *spheroidal* inclusions (aspect ratio ε). The *vertical axis* here is the axis of symmetry of TI and of the inclusions and distributions

4.2 Distribution P-tensor

In order to avoid particles overlapping, let us now consider some aspects regarding the relation between the aspect ratio ρ of the spheroid associated with the P-tensor $\mathbf{P}_d^{\rho}$, the aspect ratio ε of the inclusions and that of $\mathbf{P}_d^{\varepsilon}$ and the volume fraction ϕ_1 of the inclusion phase. Let us refer to Fig. 1 where we note that the distribution spheroid is a *security* spheroid, containing a single spheroidal inclusion, which is not intersected by any other security spheroid.

Assuming that $\mathbf{P}_s^{\varepsilon}$ and $\mathbf{P}_d^{\rho}$ are given, this construction of the composite means that there exists a maximal volume fraction associated with how much of the inclusion can fit into the security spheroid. This depends on whether $\varepsilon > \rho$ or $\varepsilon < \rho$ (see Fig. 1). Simple calculations show that when $\varepsilon > \rho$ we have $0 \le \phi_1 \le \rho^2/\varepsilon^2$, whereas if $\varepsilon < \rho$ we have $0 \le \phi_1 \le \varepsilon/\rho$.

Alternatively, suppose that the inclusion aspect ratio ε is fixed in addition to the volume fraction. This then gives a condition on the maximum ρ permitted. In particular when $\rho < \varepsilon$ we can determine that $0 \le \rho \le \varepsilon\sqrt{\phi_1} \le \varepsilon$ whereas when $\rho > \varepsilon$ we have $\varepsilon \le \rho \le \varepsilon/\phi_1$. A special case is when inclusions are spherical, so that $\varepsilon = 1$.

5 Construction of the bounds for transversely isotropic tensors

The focus of the present article is to describe a direct manner for the construction of the HS bounds for TI materials whose phases are also TI with x_3 as the axis of transverse symmetry (so that the x_1x_2 plane is the plane of isotropy). In Sect. 4.1 we have spoken of an *explicit* general construction of the HS bounds that take into account not only the inclusion shapes but also their spatial distribution, in terms of the conductive modulus tensors $\mathbf{K}^r$ and the P-tensors $\mathbf{P}_d^{\rho}$ and $\mathbf{P}_s^{\varepsilon}$ However the implementation can cause great difficulty, particularly for anisotropic phases. To simplify this issue, we shall proceed as follows. First, we observe that a second order TI tensor can be defined with respect to the tensor basis set

$$\{\mathscr{I}_{ij}^{(1)}, \mathscr{I}_{ij}^{(2)}\}, \quad \text{where} \quad \mathscr{I}_{ij}^{1} = \Theta_{ij}, \quad \mathscr{I}_{ij}^{2} = \delta_{ij} - \mathscr{I}_{ij}^{1}. \tag{5.23}$$

Hence, given (5.23) in order to define a TI conductive tensor K_{ij}^r given by (2.2), the short-hand notation $\mathbf{K}^r = (\kappa_r, \kappa_r\alpha_r)$, $r = 0, 1$, will be adopted. Observe that the contraction between the the elements of the basis tensors defined in (5.23), is given easily by

$$\mathscr{I}^{(1)}\mathscr{I}^{(1)} = \mathscr{I}^{(1)}, \quad \mathscr{I}^{(1)}\mathscr{I}^{(2)} = \mathscr{I}^{(2)}\mathscr{I}^{(1)} = 0, \quad \mathscr{I}^{(2)}\mathscr{I}^{(2)} = \mathscr{I}^{(2)}. \tag{5.24}$$

This permits us to define some basic operations over the set of second-order tensors with coefficients in $\mathbb{R}^2$.

Definition 3 Given two TI tensors K^1_{ij} and K^2_{ij} defined in short-hand notation by $\mathbf{K}^1 = (\kappa_1, \kappa_1\alpha_1)$ and $\mathbf{K}^2 = (\kappa_2, \kappa_2\alpha_2)$, using the contractions (5.24), we define the operations of addition $\mathbb{A}[\mathbf{H}^1, \mathbf{H}^2] : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, double contraction $\mathbb{C}[\mathbf{H}^1, \mathbf{H}^2] : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and inversion $\mathbb{I}[\mathbf{H}^1] : \mathbb{R}^2 \to \mathbb{R}^2$ in the following way

$$\begin{aligned} &\mathbb{A}[\mathbf{H}^1, \mathbf{H}^2] := (\kappa_1 + \kappa_2, \kappa_1\alpha_1 + \kappa_2\alpha_2), \quad \mathbb{C}[\mathbf{H}^1, \mathbf{H}^2] := (\kappa_1\kappa_2, \kappa_1\kappa_2\alpha_1\alpha_2), \\ &\quad \mathbb{I}[\mathbf{H}^1] := (1/\kappa_1, 1/(\kappa_1\alpha_1)). \end{aligned} \tag{5.25}$$

Therefore, given the explicit expressions for the HS bounds derived in Sect. 4, one can straightforwardly construct them using (5.25) and introducing the tensors coded as functions with arguments as 2-vectors.

6 Implementation and examples

In this section we illustrate the above scheme for a heterogeneous media containing aligned spheroidal inclusions of alumina embedded in a host matrix made up of aluminum. Aluminum (Al) is probably the most common matrix for metal-host composites for several reasons. It is very light, which makes it of great interest for the aerospace industry and other applications such as laptop computers. Its low cost due and low melting temperature also makes composite fabrication very economical. However, the high heat dissipation of this material is a serious problem in applications that use this material as a thermal conductor. An effective way to resolve this problem is the addition of a low filler to form a metal matrix composite. The most common filler used is silicon carbide (SiC) inclusions due to its low cost and low coefficient of thermal expansion (CTE=3.7 $10^{-6/^oC}$). In this process [17], SiC reacts with aluminum through the chemical reaction $3SiC + 4Al \to 3Si + Al_4C_3$ and the produced silicon (Si) weakens the interface between the filler and the host of the composite. Alternative, alumina (Al_2O_3) and aluminium nitride (AlN) are fillers that do not react with aluminium. We refer to [18] for more details related to the chemical reaction between these materials. Particularly, particles of alumina (Al_2O_3) are the second most popular type that are employed. Therefore, in this section we illustrate the above bounding scheme for a heterogeneous media containing aligned spheroidal inclusions of alumina embedded in a host matrix made up of aluminum. According to [17] and [18] the numerical values of thermal conductivities for aluminum and alumina are, respectively $\kappa_0 = 247\, W/Km$ (CTE= 23 $10^{-6}/^oC$) and $\kappa_1 = 20\, W/Km$ (CTE= 7 $10^{-6}/^oC$). For simplicity we compute only the transversal effective component κ_* of the composite. The HS bounds are plotted as a function of the volume fraction $\phi = \phi_1$ of the alumina inclusion phase.

In Fig. 2 we suppose that $\varepsilon = \rho$ (then $\mathbf{P}_s = \mathbf{P}_d$). It shows on the left bounds associated with *spherical* alumina inclusions uniformly distributed with *spherical*

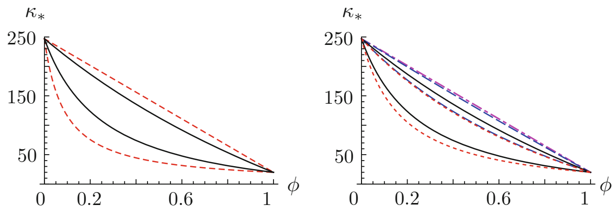

Fig. 2 $\mathbf{P}_s^{\varepsilon} = \mathbf{P}_d^{\rho}$: All the alumina inclusions have the same shape and are distributed with the same spheroidal shape. On the *left* we consider spheres and spherical statistics. The HS bounds (*solid*) and Wiener bounds (*dashed*) are plotted. On the *right* spheroidal inclusions and statistics are chosen with $\varepsilon = 1$ (*solid*), $\varepsilon = 0.1$ (*dashed*), $\varepsilon \to \infty$ (*dotted*) and $\varepsilon \to 0$ (*dot-dashed*)

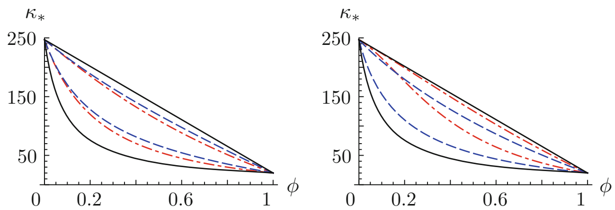

Fig. 3 $\mathbf{P}_s^{\varepsilon} \neq \mathbf{P}_d^{\rho}$. On the *left*, we take *spherical* alumina inclusions ($\varepsilon = 1$) and consider both oblate with $\rho = \sqrt{\phi}$ (*dot-dash lines*) and $\rho = \frac{1}{\phi}$ for the prolate (*dashed lines*) distributions. On the *right*, we consider the effect when the inclusion becomes a *spheroid* with (oblate and prolate) aspect ratio $\varepsilon = \phi,\ 1/\sqrt{\phi}$ (*dot-dash and dashed lines* resp.) and the distribution is *spherical* ($\rho = 1$)

symmetry in the aluminum host phase ($\rho = \varepsilon = 1$). Therefore the effective material is isotropic. We plot the Wiener bounds (dashed lines) on these effective properties together with the HS bounds (solid lines), noting the improvement of the HS bounds in particular. On the right, we consider the case of *spheroidal* alumina inclusions with general aspect ratio $\varepsilon = 1$ (solid line), $\varepsilon = 0.1$ (dashed lines) distributed with *spheroidal* symmetry. So that the effective material is TI. Limiting cases $\varepsilon \to \infty$ (dotted line) and $\varepsilon \to 0$ (dot-dashed line) corresponding to long fibre-reinforced and layered materials are also plotted.

Analogously to the elasticity context, the bounds coincide in the (layered) limit when $\varepsilon \to 0$, as can also be observed in the plots in Figs. 3 and 4. It is also seen that the transverse modulus is not affected greatly by increasing ε from unity. In fact, the fibres do not have to be particularly long before they reach this limit: $\varepsilon = \mathcal{O}(10)$ is sufficient for example.

In Fig. 3 we consider the case when the distribution spheroid has different shape to that of the inclusion shape ($\mathbf{P}_s^{\varepsilon} \neq \mathbf{P}_d^{\rho}$). On the left, we take *spherical* alumina inclusions ($\varepsilon = 1$) and consider both oblate with $\rho = \sqrt{\phi}$ (dot-dash lines) and $\rho = \frac{1}{\phi}$ for the prolate (dashed lines) distributions. On the right, we consider the effect when the inclusion becomes a *spheroid* with (oblate and prolate) aspect ratio $\varepsilon = \phi,\ 1/\sqrt{\phi}$ (dot-dash and dashed lines resp.) and the distribution is *spherical* ($\rho = 1$). It is worth observing the fact that according to (4.22), the effect of the aspect ratio of the inclusions on the effective properties of the material is larger that the corresponding one due to the distribution of spherical inclusions. Of course, each point on these curves represents

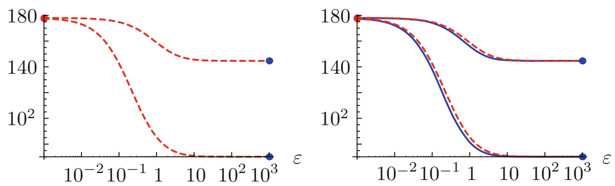

Fig. 4 Effect of the inclusion aspect ratio on the transverse effective conductivity. On the *left*: $\mathbf{P}_s^\varepsilon = \mathbf{P}_d^\rho$. On the *right*: $\mathbf{P}_s^\varepsilon \neq \mathbf{P}_d^\rho$

a different type of composite in the sense that the inclusion and the distribution has different spheroidal statistics. Wiener bounds (solid lines) are also represented. On the right of Fig. 4, we plot the influence of *spheroidal* inclusions with aspect ratio ε, possibly distinct from the aspect ratio ρ of the *spheroidal* distribution for a fixed $\phi = 0.3$. We plot two set of curves, one set corresponding to a *spheroidal* distribution with aspect ratio $\rho = \varepsilon\sqrt{0.3}$ (solid lines) and another set corresponding to the same aspect ratio as that of the inclusion $\rho = \varepsilon$ (dashed lines). Note that the plot has log-linear scaling and as should be expected the most significant effect is felt away from the limiting cases when $\varepsilon \to 0$ and $\varepsilon \to \infty$ corresponding to the layer and long fibre limits.

7 Conclusions

We have presented a straightforward mechanism for the *construction* of the HS bounds for TI composites focusing in particular on the two-phase case in the conductivity setting. The scheme takes into account microstructural information of the media through the shape and distribution of the inclusions. The explicit form of the Hill tensors for spheroidal inclusions, which is often derived from its integral form- and the definition of an appropriate TI tensor basis set is used. The associated vector notation described in Sect. 5 leads to an clear way to develop a mathematical theory that generalizes some existing formulas in the literature. We implement different constructions for a specific composite material, showing the improvement of the HS bounds over the Wiener bounds. Analogous schemes may be developed for several phases and materials of arbitrary anisotropy, although in general the corresponding Green tensor, and therefore the Hill tensor cannot be derived analytically. The mechanism proposed can be extended to the more general elasticity context by deriving the corresponding Hill tensors and the appropriate basis tensor. In this sense, future work will try to consider the construction of HS bounds for multi-phase composites, taking into account enough microstructural information to derive accurate property predictions.

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